Linear Parameter-Varying systems: a brief review and perspectives of the polytopic approach

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Abstract. This brief paper presents a review about linear parameter-varying (LPV) systems, addressing main approaches for the analysis and control designs. Moreover, some perspectives are discussed regarding its future applications. This LPV framework can be used to represent and model many different problems and systems, such as automotive systems, flight control, magnetic systems, among many others. Given the attention received by this system class, this survey aims to provide basic considerations regarding the representation of LPV systems and main mathematical definitions, including main techniques for the polytopic approach. As the Lyapunov functions remains one of the most used approaches to deal with such problems, this paper focuses on addressing their use. Finally, usual applications and problems of LPV systems are assessed in order to support the consideration of new study topics for this framework.

Keywords. LPV systems, stability analysis conditions, control design conditions.

1. Introduction

Linear parameter-varying (LPV) systems are an interesting approach to deal with many different problems [1,2]. While Linear time-varying (LTV) systems are a more general representation than the LPV one, the latter has a more concise and accessible theory as it particularizes some aspects of the former. Examples include automotive systems, flight control, wind turbines, magnetic systems and so on [3]. Therefore, the study of LPV systems became an interesting research topic in recent decades, in particular for controller designs [1].

Among the many approaches to cope with this system class, the so-called polytopic approach rises as a very interesting one. By representing the LPV system as a finite set of linear models on a convex hull, some useful techniques and methods become available to obtain analysis and design conditions [4]. Even some properties from the linear time-invariant (LTI) can be adapted, which furthers supports the LPV system studies. Lyapunov functions (LFs) are an interesting option to deal with such problems [5], as they can be applied in an organized and established procedure to obtain tractable numerical problems. Mainly, Linear Matrix Inequalities (LMIs) became a standard mathematical tool to deal with polytopic LPV system representation. LFs may be used to deal with strategies as quadratic and poly-quadratic approaches [6], both established options in this context.

Given these considerations, this brief paper aims to address some important basic definitions regarding LPV systems, focusing in the continuous-time polytopic class. The review objective is to present a comprehensive compilation of basic information regarding such topics, however assuring a satisfactory level of mathematical rigor. Besides analysis and design conditions commentaries, insights regarding works about LPV systems are also encompassed by this survey.

This paper is organized as follows. First, in Section 2 some basic definitions regarding continuous-time LPV systems are presented. Afterwards, some commentaries about stability analysis and control designs are provided. Then, Section 3 discusses some research topics regarding LPV systems. Section 4 concludes the survey.

2. LPV systems

This section will firstly present some basic definitions for the LPV systems in a polytopic form. Secondly, some of the main approaches to analyse stability and design controllers using Lyapunov function are considered.

2.1 Basic definitions

Before dealing with LPV systems, it is interesting to define the LTV system class. A state-space representation of this class can be given as

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)u(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are, respectively, the state, the input and the output vectors. If t is considered fixed, then equation (1) would rather represent a simpler LTI system.

A LPV system is LTV system whose time-varying dependency lies on a parameter. Making use of the polytopic representation, we may define the LPV system as [7]:

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B((t))u(t) \\ y(t) = C((t))u(t) \end{cases} \tag{2}$$

where it is possible to notice that all matrices now depend on the time-varying parameter $\theta(t) \in \mathbb{R}^\rho$. A comparison of these system classes can be found in Tab. 1. One may note that if the parameter is only *a priori* known, then the general LTV structure must be considered. Moreover, the presence of uncertainties on the parameter may lead to a particular system class.

Tab. 1 - Comparison of LTI, LTV and LPV systems

Parameter $\theta(t)$	*A priori* known	Real-time known	Uncertain
Constant	LTI	-	Uncertain LTI
Time-varying	LTV	LPV	Uncertain LPV

It is important to highlight that there are other structures for the LPV dependency on the time-varying parameter than the polytopic one, such as the linear fractional representation (LFR). This paper will focus in the former, given its wider popularity. In order to proper define the polytopic LPV system, it is necessary to characterize its time-varying parameter. According to the polytopic approach, an interesting formulation is as follows [5]:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{N} \mu_i(t)\, A_i x(t) + B_i u(t) \\ y(t) = \sum_{i=1}^{N} \mu_i(t)\, C_i u(t) \end{cases} \tag{3}$$

where $\mu_i(t)$ is a weighting function depending on the time-varying parameter $\theta(t)$ and $N = 2^\rho$. Given that $\theta(t)$ his limited by maximum and minimum values, then $\mu_i(t)$ belongs to the unit simplex

$$\Theta = \left\{ \mu(\theta) \in \mathbb{R}^N : \sum_{i=1}^{N} \mu_i(t) = 1, \mu_i(t) \geq 0 \right\}. \tag{4}$$

The main focus of this paper is on continuous-time LPV systems; however, for the sake of fullness, let us present an discrete-time LPV system

$$\begin{cases} x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k) \\ y(k) = C(\theta(k))u(k) \end{cases} \tag{5}$$

which is an equivalent class of the equation (2) and has compatible mathematical properties. One may note that the structure of both time frameworks is very similar. When the discretization process may lead to complications or if it is possible to model the system directly as discrete-time, then (5) becomes an attractive approach.

2.2 Stability analysis approaches

As it is well-known, while the analysis of LTI systems is straightforward for state-space systems, the same cannot be said for LPV system. LTI systems can be analysed through information as eigenvalues, frequency response, etc., but LPV systems, given its time-varying behaviour, require more sophisticated methods.

One of the most popular methods to analyse the stability of a LPV system is based on Lyapunov functions [8]. The quadratic stability analysis, which originated from more general system classes, is a possible alternative to make use of LFs. A quadratic LF $V(x(t))$ is as follows

$$V(x(t)) = x(t)^T P x(t) \tag{6}$$

and $P > 0$, which is called Lyapunov Matrix,

While simple to use to solve many problems, it is well-known that this approach can be very conservative. A straightforward strategy to overcome such drawback is the poly-quadratic approach [5], where the LF is modified such that

$$V(x(t)) = x(t)^T P(\theta(t)) x(t) \tag{7}$$

and $P(\theta(t)) > 0$.

By taking the time-varying parameter into the Lyapunov matrix P, a less conservative solution to the stability problem can be found. Therefore, an analysis condition derived with such formulation is more likely to obtain improved results than the quadratic approach in (6).

In order to proper transform Lyapunov function-based conditions in numerically tractable problems, the LMIs are a well-established tool. There are many solving programs to deal with LMI conditions for LPV systems, many of them developed to work with Matlab. Some examples include Yalmip, LmiLab and SDPT3, each one of them with its own features.

As example, a LMI condition can be given as follows. Using quadratic LF approach as in equation (6), it is possible to define that LPV system (2) is asymptotically stable if there exists a positivedefinite P such that

$$A_i^T P + P A_i < 0 \quad (8)$$

for $i = 1, \dots, N$, where state matrices A_i are obtained as in (3). This LMI condition can be promptly formulated and solved thanks to the polytopic formulation that allows the stability analysis of the LPV system by checking its polytope vertices.

Besides these two main approaches, i.e. quadratic and poly-quadratic, there are many other options to deal with LPV systems through LFs. Some main examples include polyhedral Lyapunov functions, piecewise Lyapunov functions and homogeneous Lyapunov functions [7]. Each method has its own drawbacks, mostly regarding numerical issues. Nevertheless, it is important to highlight that there are many strategies and workarounds to mitigate such drawbacks and make these approaches more useful.

2.3 Control design approaches

It is possible to adapt the stability analysis conditions previously presented to encompass controller design conditions for LPV systems. This is possible by considering a closed-loop version of the system, such that the stability of both the controller and the system can be assessed [9].

Let us first address the quadratic stability conditions from (6). A possible structure for a controller using such formulation can be given by the following control law equation:

$$u(t) = K y(t) \quad (9)$$

where $K \in \mathbb{R}^{m \times p}$ is fixed a gain to be defined such that the system is stable. This technique is known as the robust output-feedback control. By making use of the LPV system output, the controller may be obtained. The control design may have the objective to stabilize the system given its own dynamics, to force the output track a reference signal or even reject an exogenous disturbance.

As stated before, it is possible to make use of the LPV system parameter to obtain less conservative formulations. For example, let us consider the LPV controller presented in Fig. 1, where two signals are considered: time-varying parameter and output.

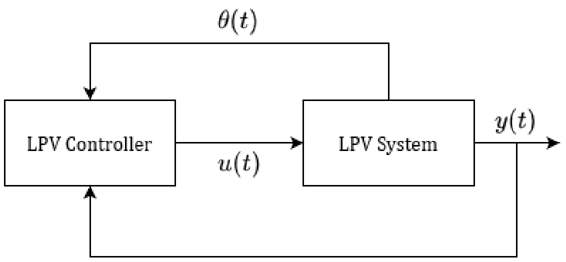

Fig. 1 - Output-based LPV controller.

By making use of the parameter-dependent LF, a possible structure for such controller can be given by the following control law equation [7]:

$$u(t) = K\big(\theta(t)\big) y(t) \quad (10)$$

where $K\big(\theta(t)\big) \in \mathbb{R}^{m \times p}$ is a gain to be defined such that the closed-loop system is stable. This technique is known as the gain-scheduled output-feedback control. Such approach is well-known to result in less conservative performance in comparison to the quadratic ones.

If the LPV system states are available, it is possible to modify the control law to

$$u(t) = K\big(\theta(t)\big) x(t) \quad (11)$$

where $K\big(\theta(t)\big) \in \mathbb{R}^{n \times p}$ is also a gain to be defined such that the system is stable. This technique is known as the gain-scheduled state-feedback control. This approach depends on the availability of the systems states or on the existence of a state observer to adequately estimate them.

Besides guaranteeing the closed-loop system stability, a controller design may also encompass other features, including performance indexes as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ criteria, uncertainty margins, robustness to faults, etc. All these considerations may be included in design conditions, as well as analysis conditions, such as the ones defined in terms of LMIs.

3. Present studies and future trends

This section begins considering some of the more popular research topics regarding LPV system. Afterwards, some commentaries regarding future works are presented.

3.1 Present researches

Nowadays, LPV systems are considered to deal with many different problems. Some of the more popular applications include: fault-tolerant strategies, including both detection and control structures [10]; non-linear and time-varying systems modelling and control [2]; data-based applications, considering statistical data to improve the application of LPV systems [11]; artificial intelligence (AI) approaches [12], combining neural networks and/or fuzzy systems to more traditional theories to obtain new results.

3.2 Future researches

Given the present studies, it is possible to consider which topics could be potential researches for LPV systems. While such opportunities already have some interesting works, certainly they are very important and trending topics regarding this area in particular and could use more researches. The main ideas include: modelling comparisons, where the many different LPV approaches, such as polytopic and LFR ones, could be proper compared under certain conditions to better understand each one and its features; complexity studies and reduction techniques, as some LPV systems may result in numerically prohibitive conditions and could benefit from such strategies; improved integration with data- and AI-based approaches, as both methods are in constant improvement in industrial and academic fields.

4. Conclusion

A brief review on LPV systems was presented by this paper. Focusing in the continuous-time case, this survey carried out the basic definitions regarding this system class using the polytopic approach. Considerations regarding the stability analysis of LPV systems were addressed, as well as some information about controller designs. Moreover, a discussion regarding present and future LPV system research topics were provided. One may note that, given the extent of the main topic, many considerations and interesting information were not addressed by this survey. Nevertheless, the provided concepts and theories arise as some of the most important regarding LPV systems. Future works may investigate and present further considerations regarding this relevant topic.

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